A Fast Hadamard Transform for Signals with Sub-linear Sparsity

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October 3, 2013

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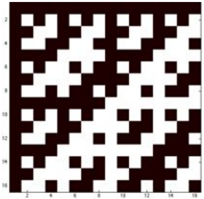
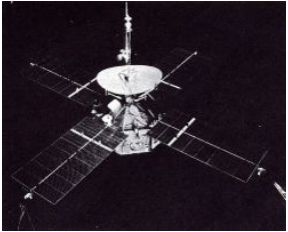
Why the Hadamard transform ?

- **Historically**, low computation approximation to DFT.
- **Coding**, 1969 *Mariner* Mars probe.
- **Communication**, orthogonal codes in WCDMA.
- **Compressed sensing**, maximally incoherent with Dirac basis.
- **Spectroscopy**, design of instruments with lower noise.



16 $\times$ 16 Hadamard matrix



Mariner probe

Related work

Ghazi et al., 2013. (Next talk!)

- Spectrum bucketing through downsampling.
- Two-dimensional sparse DFT.
- Constant probability of failure.

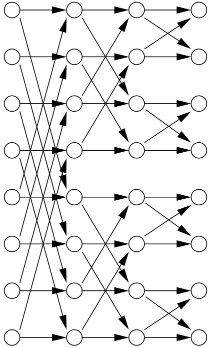
Pawar & Ramchandran, 2013.

- Spectrum bucketing through downsampling.
- One-dimensional sparse DFT.
- Length is power of small co-prime numbers.
- Probability of failure asymptotically vanishing.

Fast Hadamard transform

- Butterfly structure similar to FFT.
- Time complexity $O(N \log_2 N)$.
- Sample complexity N.
- \+ Universal, i.e. works for all signals.
- – Does not exploit signal structure (e.g. sparsity).

Can we do better ?



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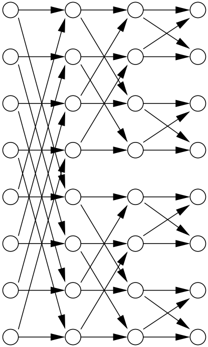

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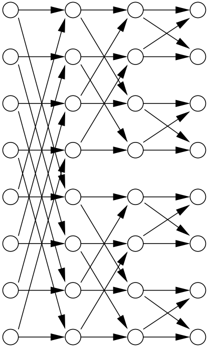

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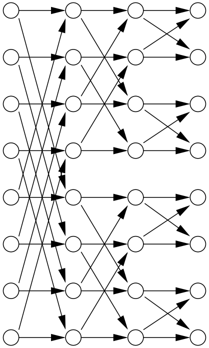

Contribution: Sparse fast Hadamard transform

Assumptions

- The signal is exaclty K-sparse in the transform domain.
- Sub-linear sparsity regime $K = O(N^{\alpha})$, $0 < \alpha < 1$.
- Support of the signal is uniformly random.

Contribution

An algorithm computing the K non-zero coefficients with:

- Time complexity $O(K \log_2 K \log_2 \frac{N}{K})$.
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Outline

1. Sparse FHT algorithm
2. Analysis of probability of failure
3. Empirical results

Another look at the Hadamard transform

- Consider indices of $x \in \mathbb{R}^N$, $N = 2^n$.
- Take the binary expansion of indices.
- Represent signal on hypercube.
- Take DFT in every direction.

$$\mathcal{I} = \{0, \ldots, 2^3 - 1\}$$

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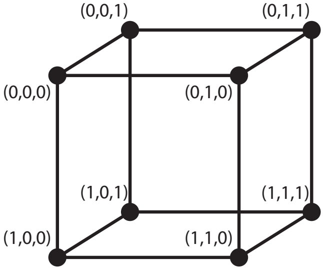

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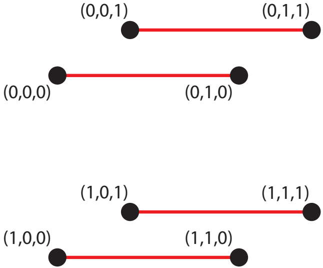

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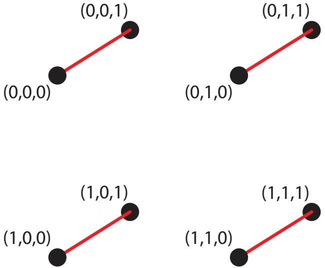

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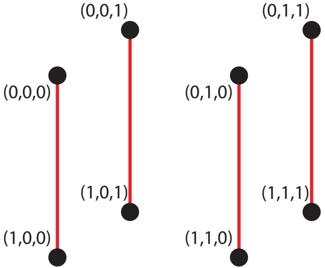

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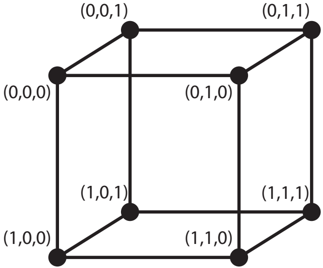

$$X_{k_0,\ldots,k_{n-1}} = \sum_{m_0=0}^{1} \cdots \sum_{m_{n-1}=0}^{1} (-1)^{k_0 m_0 + \cdots + k_{n-1} m_{n-1}} x_{m_0,\ldots,m_{n-1}},$$

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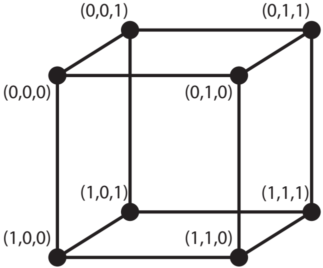

$$X_k = \sum_{m \in \mathbb{F}_2^n} (-1)^{\langle k\,,\,m\rangle} x_m, \qquad k, m \in \mathbb{F}_2^n, \quad \langle k\,,\,m\rangle = \sum_{i=0}^{n-1} k_i\, m_i.$$

Treat indices as binary vectors.

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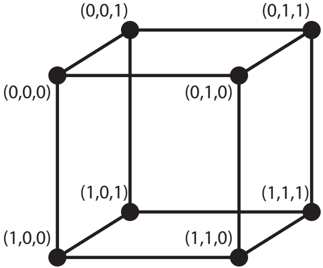

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Hadamard property I: downsampling/aliasing

Given $B = 2^b$, a divider of $N = 2^n$, and $\mathcal{H} \in \mathbb{F}_2^{b\times n}$, where rows of $\mathcal{H}$ are a subset of rows of identity matrix,

$$x_{\mathcal{H}^T m} \overset{WHT}{\longleftrightarrow} \sum_{i \in \mathcal{N}(\mathcal{H})} X_{\mathcal{H}^T k + i}, \quad m, k \in \mathbb{F}_2^b.$$

e.g. $\mathcal{H} = \left[0_{b\times(n-b)} \; I_b \right]$ selects the b high order bits.

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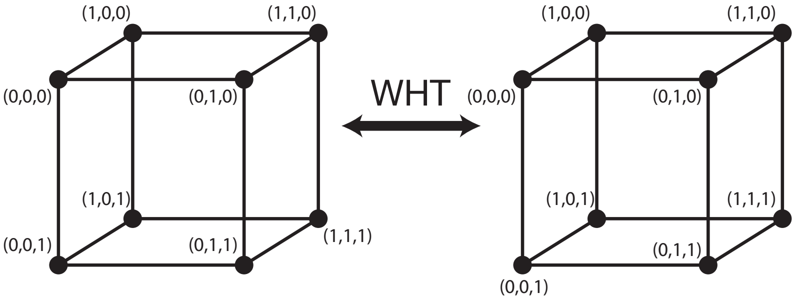

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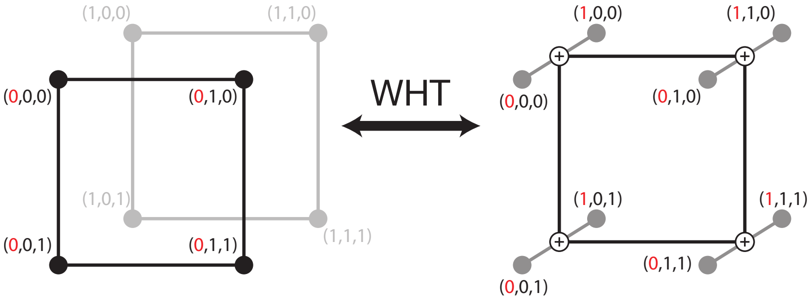
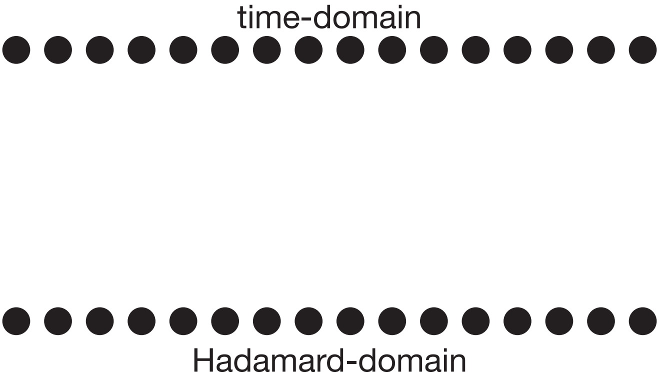


Aliasing induced bipartite graph



- Downsampling induces an aliasing pattern.
- Different downsamplings produce different patterns.

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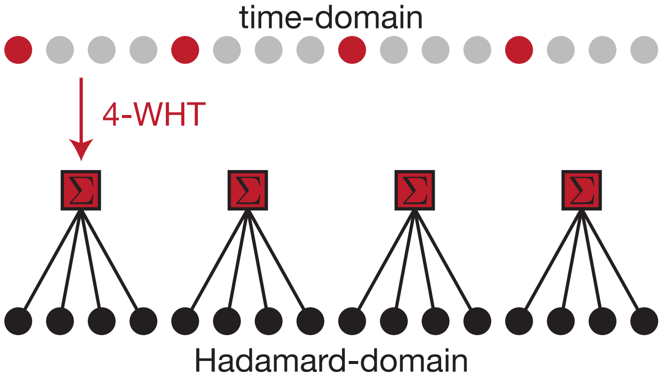

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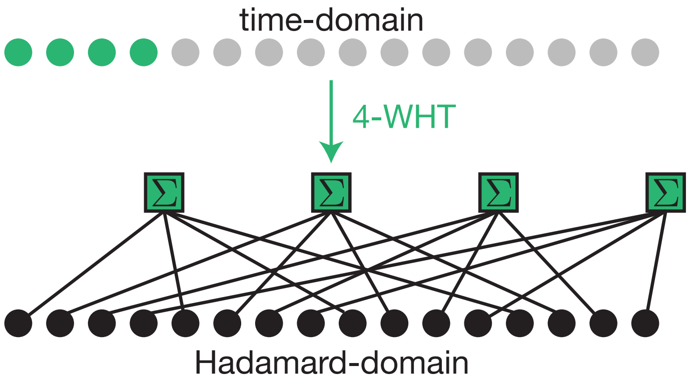

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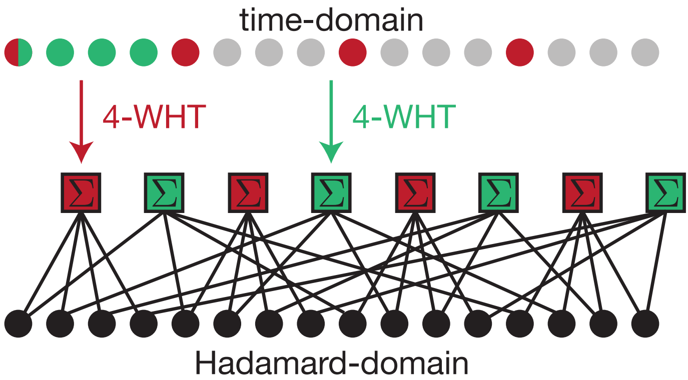

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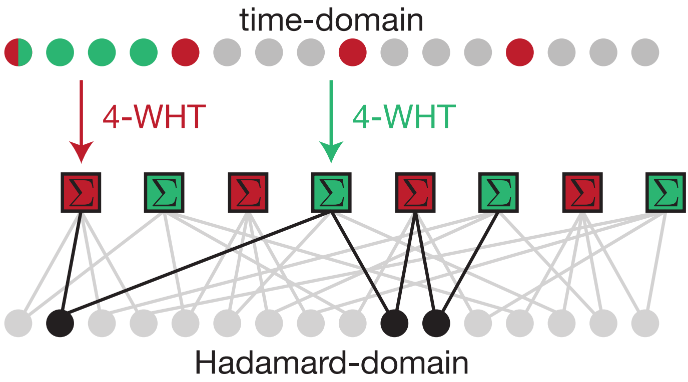

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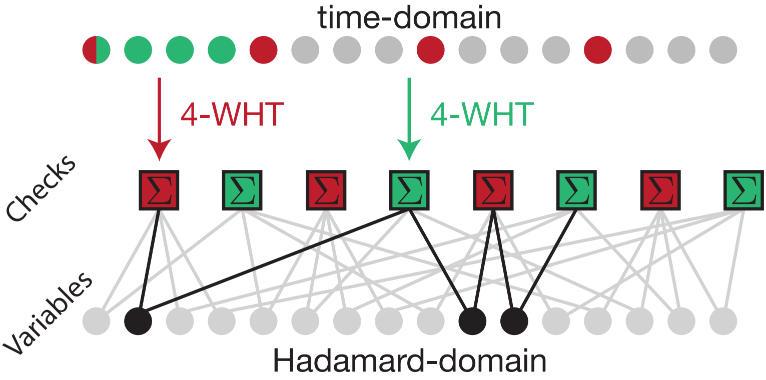

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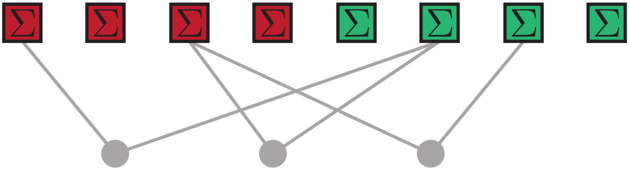
Genie-aided peeling decoder

A genie indicates us

- if a check is connected to only one variable (singleton),
- in that case, the genie also gives the index of that variable.



Peeling decoder algorithm:

1. Find a singleton check: $\{X_1, X_8, X_{11}\}$
2. Peel it off.
3. Repeat until nothing left.

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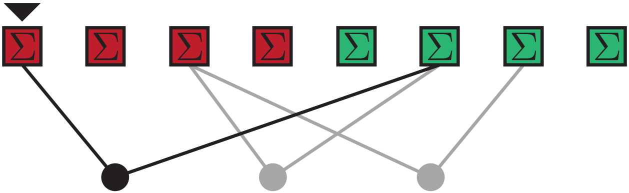

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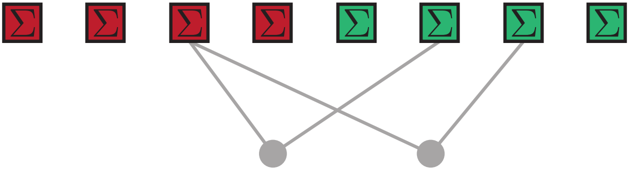

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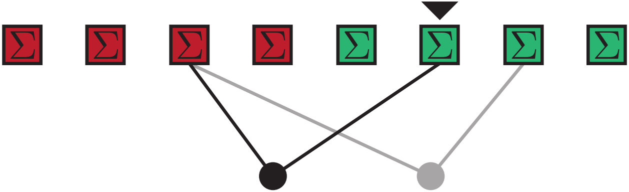

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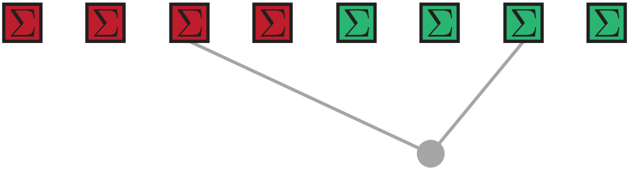

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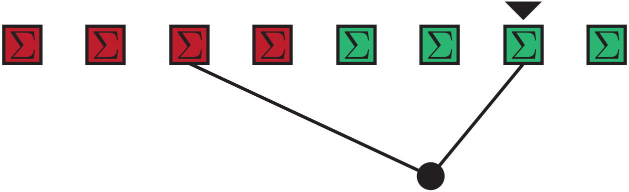

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Hadamard property II: shift/modulation

Theorem (shift/modulation)

Given $p \in \mathbb{F}_2^n$,

$$x_{m+p} \overset{WHT}{\longleftrightarrow} X_k \, (-1)^{\langle p \, , \, k \rangle}.$$

Consequence

The signal can be modulated in frequency by manipulating the time-domain samples.

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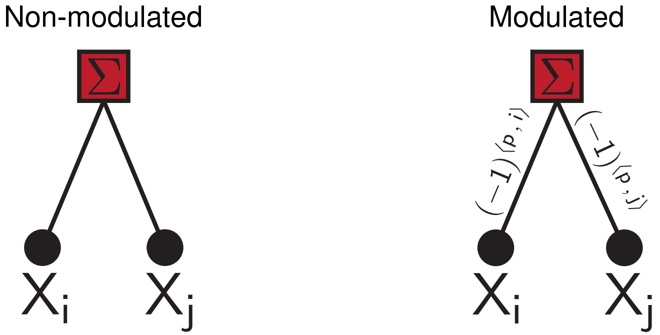
How to construct the Genie

Non-modulated | Modulated

- Collision: if ≥ 2 variables connected to same check.
- $\frac{X_i(-1)^{\langle p\,,\,i\rangle}+X_j(-1)^{\langle p\,,\,j\rangle}}{X_i+X_j} \neq \pm 1$, (mild assumption on distribution of X).

How to construct the Genie



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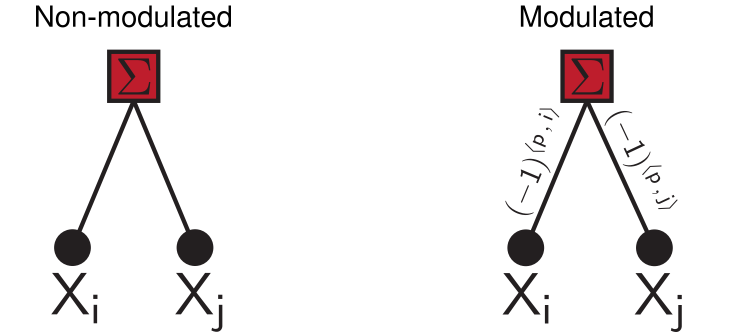

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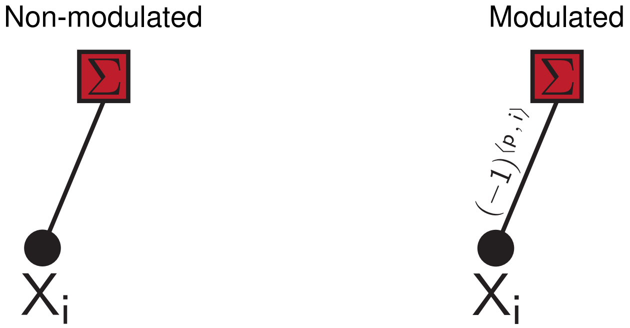

- Singleton: only one variable connected to check.
- $\frac{X_i(-1)^{\langle p,i \rangle}}{X_i} = (-1)^{\langle p,i \rangle} = \pm 1$. We can know $\langle p, i \rangle$!
- $O(\log_2 \frac{N}{K})$ measurements sufficient to recover index i, (dimension of null-space of downsampling matrix $\mathcal{H}$).

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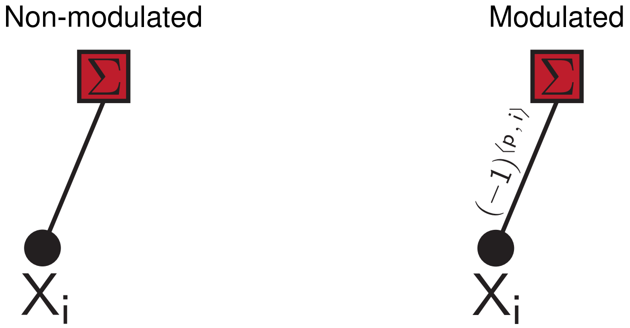

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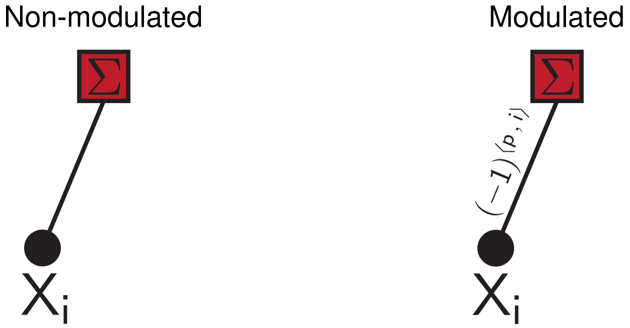

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Sparse fast Hadamard transform

Algorithm

1. Set number of checks per downsampling $B = O(K)$.
2. Choose C downsampling matrices $\mathcal{H}_1, \dots, \mathcal{H}_C$.
3. Compute $C(\log_2 N/K + 1)$ size-K fast Hadamard transform, each takes $O(K \log_2 K)$.
4. Decode non-zero coefficients using peeling decoder.

Performance

- Time complexity – $O(K \log_2 K \log_2 N/K)$.
- Sample complexity – $O(K \log_2 \frac{N}{K})$.
- How to construct $\mathcal{H}_1, \dots, \mathcal{H}_C$? Probability of success ?

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Very sparse regime

Setting

- $K = O(N^\alpha)$, $0 < \alpha < 1/3$.
- Uniformly random support.
- Study asymptotic probability of failure as $n \to \infty$.

Downsampling matrices construction

- Achieves values $\alpha = \frac{1}{C}$, i.e. $b = \frac{n}{C}$.
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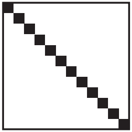

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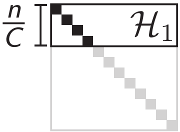

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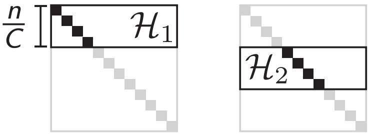

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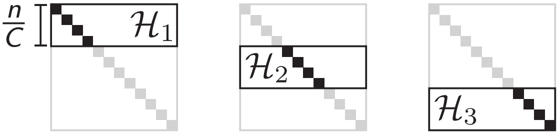

Balls-and-bins model

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- Theorem: Both constructions are equivalent.
  Proof: By construction, all rows of $\mathcal{H}_i$ are linearly independent. □
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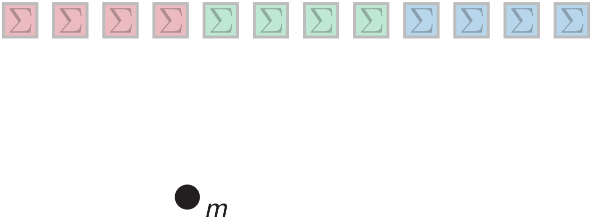

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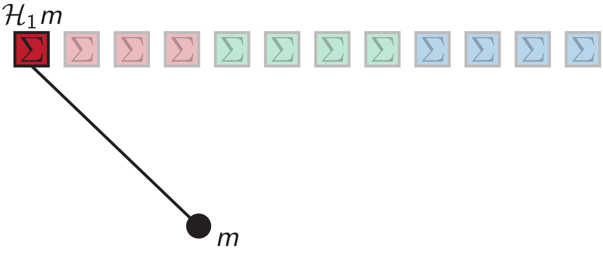

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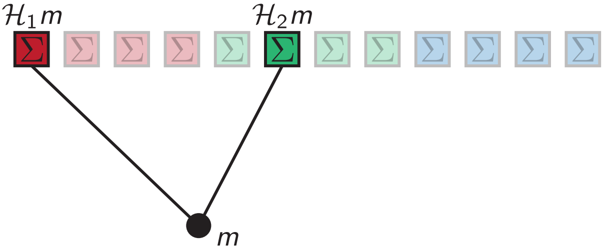

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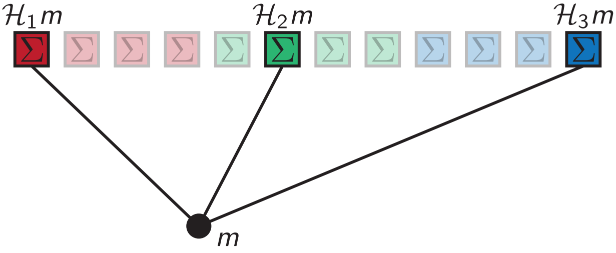

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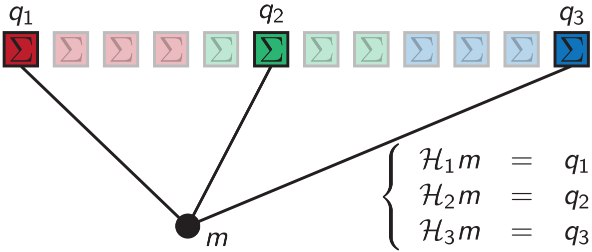

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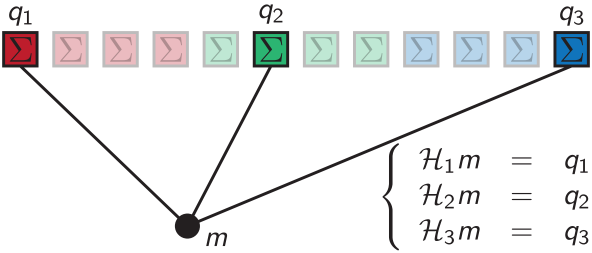

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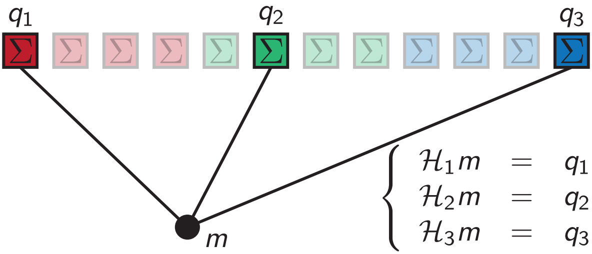

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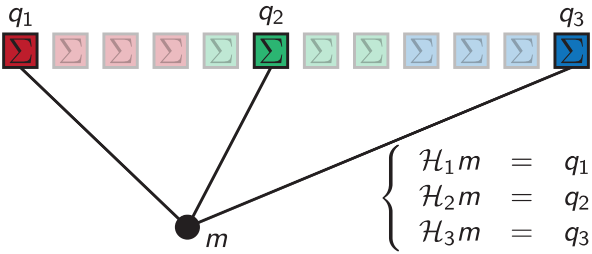

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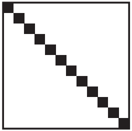

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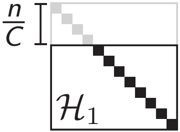

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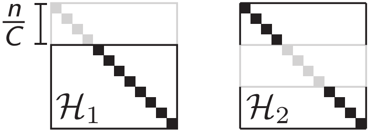

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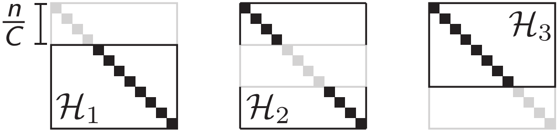

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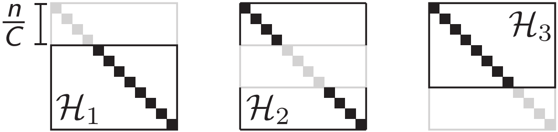

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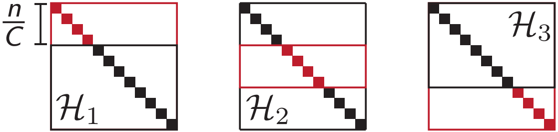

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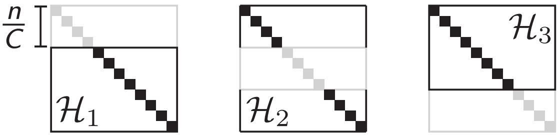

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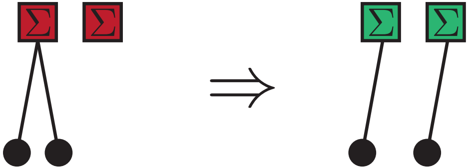
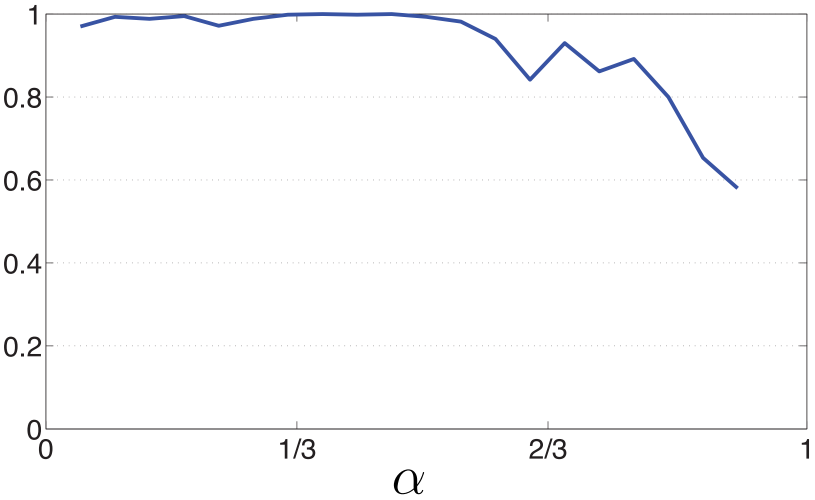
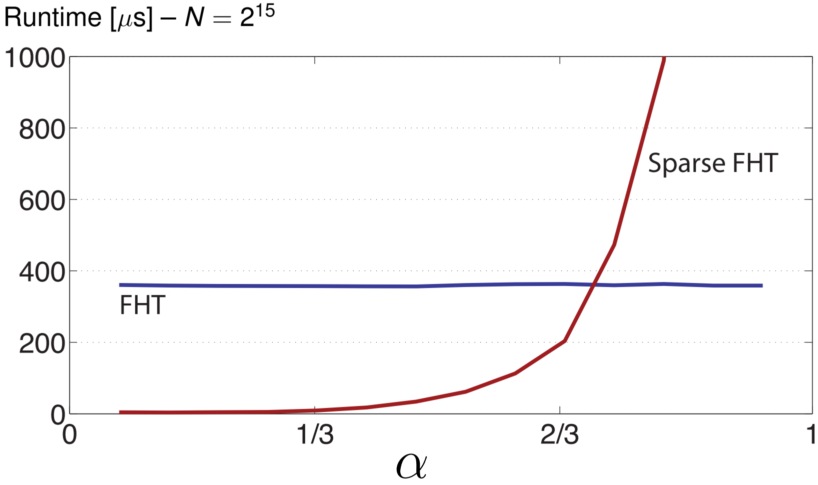


SparseFHT – Probability of success

Probability of success - $N = 2^{22}$



SparseFHT vs. FHT



Conclusion

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- Time complexity $O(K \log_2 K \log_2 \frac{N}{K})$.
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- Probability of success asymptotically equal to 1.

What's next ?

- Investigate noisy case.

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Thanks for your attention!

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FÉDÉRALE DE LAUSANNE

Code and figures available at

http://lcav.epfl.ch/page-99903.html